called the n^{th} factor of the product. The following symbols are used to denote the product defined by (1)

$$u_1 u_2 \cdots u_n, \quad \prod_{n=1}^{\infty} u_n$$

Note: The symbol $\prod_{n=N+1}^{\infty} u_n$ means $\prod_{n=1}^{\infty} u_{N+n}$

Def: 8.50

Given an infinite product $\prod_{n=1}^{\infty} u_n$

let $P_n = \prod_{k=1}^{n} u_k$

a) If infinitely many factors u_n are zero, we say that product diverges to zero.

b) If no factor u_n is zero, we say the product converges if there exists a number $p \neq 0$ such that $\{P_n\}$ converges to p.

In this case, p is called the value of the product and we write $p = \prod_{n=1}^{\infty} u_n$

If $\{P_n\}$ converges to zero, we say the product diverges to zero.

c) If there exists an N such that $n > N$ implies $u_n \neq 0$, we say $\prod_{n=1}^{\infty} u_n$ converges, provided that $\prod_{n=N+1}^{\infty} u_n$ converges as described in (b). In this case the value of the product $\prod_{n=1}^{\infty} u_n$ is $u_1, u_2 \cdots u_N \prod_{n=N+1}^{\infty} u_n$. is

$$u_1 . u_2 \cdots u_N \prod_{n=N+1}^{\infty} u_n .$$

d) $\prod_{n=1}^{\infty} u_n$ is called divergent if it does not converge as described in (b) (or) (c)

Note:

The value of a convergent infinite product can be zero. But this happens iff, a finite number of factor are zero; the convergence of an infinite product or not affected by inserting ~~or~~ removing a finite number of factors, zero or not

Example : (1)

P.T $\prod_{n=1}^{\infty} (1+ 1/n)$ diverges.

Consider the partial product.

$$P_n = (1+1/1)(1+1/2) \cdots (1+1/n)$$

$$= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n}$$

$$= n+1$$

Since $\{P_n\} = \{n+1\}$ diverges

$\therefore \prod_{n=1}^{\infty} (1+\frac{1}{n})$ diverges

Example : (2)

S.T $\prod_{n=2}^{\infty} (1-1/n)$ diverges

Consider the partial product

$$P_n = (1-1/2)(1-1/3) \cdots (1-1/n)$$

$$= 1/2 \cdot 2/3 \cdot 3/4 \cdots n-1/n$$

$$= 1/n$$

$\{P_n\} = \{1/n\} \longrightarrow 0$ as $n \longrightarrow \infty$

$\prod (1-1/n)$ diverges to '0'

$\prod_{n=2} (1-1/n)$ diverges

Theorem : 8.51

<u>Cauchy Condition for Convergence of the products</u>

The infinite product $\prod_{n=1}^{\infty} u_n$ converges if and only if for every $\epsilon > 0$ there exists an N such that $n > N$ implies

$$|\{u_{n+1}, u_{n+2}, \cdots u_{n+k}\}\} - 1| < \epsilon \quad \text{for } k = 1, 2, 3 \ldots$$

proof :

Necessary part :

Assume that the infinite product $\prod_{n=1}^{\infty} u_n$ converges

TPT: for every $\epsilon > 0$ there exists an N such that

$n > N \Rightarrow |u_{n+1} \cdot u_{n+2} \cdot \ldots u_{n+k} - 1| < \epsilon$, for $k = 1, 2, 3$

We know that a convergent product can have only finite number of zero factors.

Therefore discarding a few terms if necessary we can assume that no u_n is zero.

Let $P_n = u_1 \cdot u_2 \cdot \ldots \cdot u_n$

and $\lim_{n \to \infty} P_n = P$

Since $\prod u_n$ is convergent, by definition

That is $\{P_n\}$ converges to a non zero numbers P

Hence there exists an $M > 0$ such that

$|P_n| > M$

We know that a sequence of real numbers is convergent if and only if it is a cauchy sequence

So $\{P_n\}$ satisfies the cauchy condition for sequence.

Hence, given $\epsilon > 0$ there exists N such that

$|P_{n+k} - P_n| < M \cdot \epsilon$ for $n > N$ and $k = 1, 2, \ldots$

$\Rightarrow |u_1, u_2, \ldots u_{n+k} - u_1, u_2, \ldots u_n| < \epsilon M$

$\Rightarrow |(u_1, u_2 \ldots u_n) \{(u_{n+1}, u_{n+2}, \ldots u_{n+k}) - 1\}| < \epsilon M$

$\Rightarrow |P_n \{(u_{n+1} \cdot u_{n+2}, \ldots u_{n+k}) - 1\}| < \epsilon M$

$\Rightarrow |P_n| \, |u_{n+1}, u_{n+2}, \ldots u_{n+k} - 1| < \epsilon M$

$\Rightarrow |u_{n+1}, u_{n+2}, \ldots u_{n+k} - 1| < \frac{\epsilon}{|P_n|} M$ $\quad [\because |P_n| > M$

$\Rightarrow |u_{n+1}, u_{n+2}, \ldots u_{n+k} - 1| < \epsilon \cdot \frac{1}{M} \cdot M$ $\quad \Rightarrow \frac{1}{|P_n|} < \frac{1}{M}]$

$\Rightarrow |u_{n+1}, u_{n+2}, \ldots u_{n+k} - 1| < \epsilon$

for every $\epsilon > 0$ there exists N such that $n > N$.

$\Rightarrow |u_{n+1}, u_{n+2}, \ldots u_{n+k} - 1| < \epsilon$, $k = 1, 2 \ldots$

Sufficient part :

Assume that, for every $\epsilon > 0$ there exist N

such that $n > N$

$\Rightarrow |u_{n+1} \cdot u_{n+2} \cdots u_{n+k} - 1| < \epsilon \quad , k = 1, 2 \ldots$

TPT The infinite product $\prod_{n=1}^{\infty} u_n$ converges

If $u_n = 0$, for any $n > N$ then (1) becomes

$\Rightarrow |0 - 1| < \epsilon$

$\Rightarrow |0 - 1| < \epsilon$

$\Rightarrow 1 < \epsilon$

This is impossible for $\epsilon < 1$. Hence $n > N$ implies $u_n \neq 0$

Take, $\epsilon = 1/2$ in (1). Let N_0 be the corresponding N, then (1)

$\Rightarrow |u_{N_0+1} \cdot u_{N_0+2} \cdots u_{N_0+k} - 1| < 1/2$ for $k = 1, 2 \ldots \longrightarrow (2)$

Let $q_n = u_{N_0+1} \cdot u_{N_0+2} \cdots u_n$ if $n > N_0$

If $n > N_0$, then $n = N_0 + k$ for some k.

Hence (2) $\Rightarrow |q_n - 1| = |u_{N_0+1} \cdot u_{N_0+2} \cdots u_{N_0+k} - 1| < 1/2$

i.e.) $|q_n - 1| < 1/2$

$\Rightarrow -1/2 < q_n - 1 < 1/2$

$\Rightarrow -1/2 + 1 < q_n < 1/2 + 1$

$\Rightarrow 1/2 < q_n < 3/2$ for $i > N_0 \longrightarrow (3)$

If $\{q_n\}$ converges, it cannot converges to zero

TPT $\{q_n\}$ converges,

Let $\epsilon > 0$ be arbitrary

for $n > N$,

(1) $\Rightarrow |u_{n+1} \cdot u_{n+2} \cdots u_{n+k} - 1| < \epsilon \quad , k = 1, 2 \ldots$

$$\Rightarrow \left| \frac{u_{N_0+1} \; u_{N_0+2} \cdots u_n \; u_{n+1} \; u_{n+2} \cdots u_{n+k}}{u_{N_0+1} \; u_{N_0+2} \cdots u_n} - 1 \right| < \epsilon$$

$$\Rightarrow \left| \frac{q_{n+k}}{q_n} - 1 \right| < \epsilon$$

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$$\Rightarrow \left| \frac{q_{n+k} - q_n}{q_n} \right| < \epsilon$$

$$\Rightarrow |q_{n+k} - q_n| < |q_n| \epsilon$$

$$< \frac{3}{2}\epsilon \quad [\text{by } 2]$$

$$\Rightarrow |q_{n+k} - q_n| < \frac{3\epsilon}{2} \quad \forall\, n > N$$

$\{q_n\}$ satisfies the cauchy condition for sequences and hence convergent

Since, q_n is partial product of the product u_{N_0+1}, u_{N_0+2} $\ldots = \prod_{n=N_0+1}^{\infty} u_n$ and $\{q_n\}$ converges non zero number

$$\prod_{n=N_0+1}^{\infty} u_n \text{ converges}$$

and hence the product $\prod u_n$ converges

Note : Taking $k=1$ in (1) we have

for $\epsilon > 0$ there exists N such that

$$n > N \Rightarrow |u_{n+1} - 1| < \epsilon$$

$$\Rightarrow \lim_{n \to \infty} u_n = 1$$

convergence of $\prod u_n$ implies that

$$\lim_{n \to \infty} u_n = 1$$

Then convergence of $\prod(1+a_n)$ implies $\lim_{n\to\infty} a_n = 0$.

Theorem: 8.52

Assume that each $a_n > 0$ then the product $\prod(1+a_n)$ converges if and only if the series $\sum a_n$ converges.

proof: To prove this first we shall prove the following inequality

$$1 + x \leq e^x \text{ for all real } x$$

we need to prove the inequality only for $x \geq 0$

The function is differentiable and continuous in the interval $[0, x]$

Applying mean value theorem for the function e^x on $[0, x]$

we can find $x_0 \in (0, x)$ such that

$$\frac{e^x - e^0}{x - 0} = e^{x_0}$$

$$\Rightarrow e^x - 1 = x e^{x_0}$$

$$\Rightarrow e^x - 1 \geq x \qquad [\because e^{x_0} \geq 1]$$

$$\Rightarrow 1 + x \leq e^x \longrightarrow (1)$$

$x_0 > 0$
$e^{x_0} > e^0$
$e^{x_0} > 1$

Now we consider the n^{th} partial sum of the series $\sum a_n$

$S_n = a_1 + a_2 + \cdots + a_n$ and we consider the n^{th} partial product of $\prod (1 + a_n)$

$P_n = (1 + a_1)(1 + a_2) \cdots (1 + a_n)$

To prove the theorem we need P.T $\{S_n\}$ converges iff $\{P_n\}$ converges.

Since $a_n > 0$, $P_n \geq 1$ and both the sequence $\{S_n\}$ and $\{P_n\}$ are increasing

We know that an increasing sequence is convergent if it is bounded above

Hence to prove the theorem it suffices to P.T $\{S_n\}$ is bounded iff $\{P_n\}$ is bounded.

We have,

$$P_n = (1 + a_1)(1 + a_2) \cdots (1 + a_n)$$

$$= 1 + a_1 + a_2 + \cdots + a_n + a_1 a_2 + a_2 a_3 + \cdots + a_1 a_2 \cdots a_n$$

$$> a_1 + a_2 + \cdots + a_n = S_n$$

$$\Rightarrow P_n > S_n \longrightarrow (2)$$

Let $x = a_k$ in (1)

$\Rightarrow 1 + a_k \le e^{a_k}, \quad k = 1, 2, \ldots n$

we have,

$$1 + a_1 \le e^{a_1}, \quad 1 + a_2 \le e^{a_2}, \ldots, 1 + a_n \le e^{a_n}$$

$$\therefore P_n = (1+a_1)(1+a_2)\ldots(1+a_n)$$

$$\le e^{a_1} e^{a_2} \ldots e^{a_n}$$

$$= e^{a_1 + a_2 + \ldots + a_n}$$

$$= e^{S_n}$$

$$P_n \le e^{S_n} \longrightarrow (3)$$

Hence from (2) and (3) we have $\{S_n\}$ is bounded iff $\{P_n\}$ is bounded.

Also, the product $\prod(1+a_n)$ converges iff the series $\sum a_n$ converges.

<u>Note</u>:- $\{P_n\}$ cannot converge to 0 since each $P_n \ge 1$ and also $P_n \longrightarrow +\infty$ if $S_n \longrightarrow +\infty$.

Definition : 8.53

The product $\prod(1+u_n)$ is said to converges absolutely if $\prod(1+|a_n|)$ converges.

Theorem : 5.3

Q: Show that absolute converges of infinite product $\prod(1+a_n)$ implies convergence.

Proof: Given that the product $\prod(1+a_n)$ converges absolutely

To prove :- $\prod(1+a_n)$ converges.

We have to P.T $\prod(1+a_n)$ satisfies cauchy condition

(ie) TPT : for any given $\epsilon > 0$, we can find N such that

$$|u_{n+1} u_{n+2} \ldots u_{n+k} - 1| < \epsilon \quad \forall\ n > N \quad k = 1, 2, \ldots$$

$$|(1+a_{n+1})(1+a_{n+2}) \cdot \ldots \cdot (1+a_{n+k}) - 1| < \epsilon$$

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Since $\prod(1+a_n)$ converges absolutely

$\prod(1+|a_n|)$ converges

By applying cauchy condition for the convergence of the infinite product we have,

For any given $\epsilon>0$ there exists N such that

$$|(1+|a_{n+1}|)(1+|a_{n+2}|)\cdots(1+|a_{n+k}|)-1|<\epsilon$$

ie) $(1+|a_{n+1}|)(1+|a_{n+2}|)\cdots(1+|a_{n+k}|)-1<\epsilon \longrightarrow (1)$

Consider,

$$|(1+a_{n+1})(1+a_{n+2})\cdots(1+a_{n+k})-1|$$

$$= |1+a_{n+1}+a_{n+2}+\cdots+a_{n+k}+a_{n+1}a_{n+2}+\cdots+a_{n+1}a_{n+2}\cdots a_{n+k}-1|$$

$$\leq |a_{n+1}|+|a_{n+2}|+\cdots+|a_{n+k}|+|a_{n+1}||a_{n+2}|+\cdots|a_{n+k}| + \cdots + |a_{n+1}||a_{n+2}|\cdots|a_{n+k}|$$

$$= 1+|a_{n+1}|+|a_{n+2}|+\cdots+|a_{n+k}|+|a_{n+1}||a_{n+2}|+\cdots+ |a_{n+1}||a_{n+2}|\cdots|a_{n+k}|-1$$

$$= (1+|a_{n+1}|)(1+|a_{n+2}|)\cdots(1+|a_{n+k}|)-1$$

$<\epsilon$ by (1)

$\Rightarrow$ for any given $\epsilon>0$ there exists for such that

$$|(1+a_{n+1})(1+a_{n+2})\cdots(1+a_{n+k})-1|<\epsilon, \ \forall\ n>N,$$
$$k=1,2,\ldots$$

Thus the product $\prod(1+a_n)$ satisfies the cauchy condition for products.

$\therefore \prod(1+a_n)$ is convergent.

Note:

From theorems 8.52 and 8.54 we see that $\prod(1+a_n)$ converges absolutely if and only if $\sum a_n$ converges absolutely.

Theorem: 8.55

Assume that each $a_n \geq 0$ then the product $\prod(1-a_n)$ converges if and only if the series $\sum a_n$ converges.

Proof: Given that $a_n \geq 0$

Necessary Part:

Assume that the series $\sum a_n$ converges

T.P.T : $\prod(1-a_n)$ converges.

Since $a_n \geq 0$ and $\sum a_n$ converges, $\sum |a_n|$ converges

$\therefore$ $\sum a_n$ converges absolutely

The product $\prod(1+|-a_n|)$ converges (by the note)

The product $\prod(1+(-a_n))$ converges absolutely

hence the product $\prod(1-a_n)$ converges

Sufficient Part:

Assume that $\prod(1-a_n)$ converges

T.P.T : $\sum a_n$ converges.

Assume that the contrary that $\sum a_n$ diverges

If $\{a_n\}$ does not converge to zero then $\prod(1-a_n)$ diverges. (by the note under 8.51)

This contradicts our assumption that $\prod(1-a_n)$ converges.

$\therefore$ We can assume that $a_n \to 0$ as $n \to \infty$

(ie) $\lim_{n\to\infty} a_n = 0$.

for any given $\epsilon = 1/2$ there exists N such that

$$|a_n| < 1/2 \quad \forall \; n > N$$

$$\Rightarrow -1/2 < a_n < 1/2$$

Since removing a finite number of factors does not affect the convergence of a product, without loss of generality we can assume that,

each $a_n < 1/2$

$\therefore -a_n > -1/2$

and hence $1 - a_n > 1 - 1/2$

$\Rightarrow 1 - a_n > 1/2$

$\Rightarrow 1 - a_n \neq 0 \ \forall n.$

let $P_n = (1-a_1)(1-a_2) \cdots \cdots (1-a_n)$

$q_n = (1+a_1)(1+a_2) \cdots \cdots (1+a_n)$

Since we have $(1+a_k)(1-a_k) = 1 - a_k^2 \leq 1$

$\Rightarrow (1+a_k)(1-a_k) \leq 1$

$\Rightarrow (1-a_k) \leq \frac{1}{(1+a_k)}$

Taking $k = 1, 2, \ldots \ldots n$ and then multiplying we have

$$(1-a_1)(1-a_2) \cdots \cdots (1-a_n) \leq \frac{1}{(1+a_1)(1+a_2) \cdots (1+a_n)}$$

$\Rightarrow P_n \leq \frac{1}{q_n} \longrightarrow (1)$

let $s_n = a_1 + a_2 + \cdots \cdots + a_n$

Then since $\sum a_n$ diverges

$\lim_{n \to \infty} s_n = \infty$

$\Rightarrow q_n \longrightarrow \infty$

from (1) $P_n \to 0$ as $n \to \infty$

(ie) $\prod (1-a_n)$ diverges to 0.

This contradiction our I assumption that $\prod (1+a_n)$ converges

$\therefore$ our second assumption is wrong.

Thus $\sum a_n$ converges.